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General solution of a three-body problem in the plane

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Abstract
We provide and discuss the *general* solution of a (Hamiltonian) three-body problem in the plane, characterized by Newtonian equations of motion with rotation- and translation-invariant velocity-dependent one-body and two-body forces. The model features a (nonnegative) real parameter ω: when it does not vanish, *all* solutions are *completely periodic* with period $T = 2\pi/\omega$; when it vanishes, both unbounded and confined motions are possible with a rather rich phenomenology of possible behaviour in the latter case, including completely periodic motions and limit cycles.

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1. Introduction and main result

Recently much interest has been focused [1–3] on the N-body problem in the plane characterized by the following Newtonian equations of motion:

$$\ddot{\vec{r}}_n = \omega\hat{k}\wedge\dot{\vec{r}}_n + 2\sum_{m=1,m\neq n}^{N}(r_{nm})^{-2}(\alpha_{nm}+\alpha'_{nm}\hat{k}\wedge)[\dot{\vec{r}}_n(\dot{\vec{r}}_m\cdot\vec{r}_{nm})+\dot{\vec{r}}_m(\dot{\vec{r}}_n\cdot\vec{r}_{nm})-\vec{r}_{nm}(\dot{\vec{r}}_n\cdot\dot{\vec{r}}_m)]. \tag{1.1a}$$

Here the N 2-vectors $\vec{r}_n \equiv \vec{r}_n(t)$ identify the positions of the moving point-particles in a plane which for notational convenience is immersed in three-dimensional space, so that $\vec{r}_n \equiv (x_n, y_n, 0)$; $\hat{k}$ is the unit 3-vector orthogonal to that plane, $\hat{k} \equiv (0, 0, 1)$, so that $\hat{k}\wedge\vec{r}_n \equiv (-y_n, x_n, 0)$;

$$\vec{r}_{nm} \equiv \vec{r}_n - \vec{r}_m \qquad r^2_{nm} = r^2_n + r^2_m - 2\vec{r}_n\cdot\vec{r}_m \tag{1.1b}$$

superimposed dots denote of course time derivatives; and the parameter ω is of course *real* (indeed, without loss of generality, *nonnegative*, $\omega \geqslant 0$).

To treat this many-body problem it is generally convenient to identify the *real* 'physical' plane in which the points $\vec{r}_n \equiv (x_n, y_n, 0)$ move with the *complex* plane in which the complex numbers $z_n = x_n + \mathrm{i}y_n$ move. Indeed via this correspondence the equations of motion (1.1) take the following simpler form:

$$\ddot{z}_n - \mathrm{i}\omega\dot{z}_n = 2\sum_{m=1,m\neq n}^{N} a_{nm}\dot{z}_n\dot{z}_m/(z_n - z_m) \tag{1.2a}$$

with

$$a_{nm} = \alpha_{nm} + \mathrm{i}\alpha'_{nm}. \tag{1.2b}$$

These equations of motion are evidently translation- and rotation-invariant, and they are also Hamiltonian, provided the 'coupling constants' a_{nm} depend symmetrically on their two indices, $a_{nm} = a_{mn}$ [1–3]. When these coupling constants take special values the model exhibits a particularly simple behaviour: for instance when the a_{nm} are all unity, $a_{nm} = 1$, the equations of motion (1.1) (or equivalently (1.2)) are *integrable* and indeed *solvable* ('goldfish' model [2, 4, 5]).

In this paper we point out that, in the three-body case with the a_{nm} all equal to minus one half, $N = 3$, $a_{nm} = -1/2$, the equations of motion (1.2*a*), which then read

$$\ddot{z}_n - \mathrm{i}\omega\dot{z}_n = -\sum_{m=1,m\neq n}^{3} \dot{z}_n\dot{z}_m/(z_n - z_m) \qquad n = 1, 2, 3 \tag{1.3}$$

can as well be solved, indeed in terms of elementary functions. The likelihood that this case is exceptionally simple had already been pointed out [7], in a context that will be reviewed below (see section 3). The fact that the *general* solution of this model is *completely periodic* and indeed *isochronous* (iff the constant ω is *real* and it does not vanish) is a remarkable consequence, as noted below (in section 2); but in this paper the case when this constant ω vanishes, and the solutions display a richer phenomenology, is also discussed (see section 2). Let us indeed emphasize that the substantive contribution of this paper is to provide in completely explicit form the *general* solution of this three-body problem in the plane, not merely to point out that this solution is (if ω is *real* and does not vanish) completely periodic and indeed isochronous, which is just a consequence of the main result. (We thank a referee for suggesting that this aspect be made crystal clear by adding the last two sentences of this paragraph).

The *general* solution of these equations of motion reads

$$z_1(t) = Z(\tau) + \zeta_+(\tau) \qquad z_2(t) = Z(\tau) + \zeta_-(\tau) \qquad z_3(t) = Z(\tau) - \zeta_+(\tau) - \zeta_-(\tau) \tag{1.4}$$

$$\tau = [\exp(\mathrm{i}\omega t) - 1]/(\mathrm{i}\omega) \tag{1.5}$$

$$Z(\tau) = Z(0) + V\tau \tag{1.6}$$

$$\begin{aligned}\zeta_\pm(\tau) = {} & a\cos(2\lambda\tau + 2\alpha \pm \pi/3) + b\cos(\lambda\tau - \beta \mp \pi/3)\exp(-\sqrt{3}\lambda\tau) \\ & \pm 2(a^2/b)\sin[2(\alpha-\beta)]\cos(\lambda\tau + 2\alpha + \beta \pm \pi/6)\exp(\sqrt{3}\lambda\tau)\end{aligned} \tag{1.7}$$

$$\lambda = V/\{2\sqrt{3}a\sin[2(\alpha+\beta) + \pi/6]\}. \tag{1.8}$$

Here $Z(0), V, a, b, \alpha, \beta$ are six arbitrary (complex) constants.

Formulae (1.4)–(1.8) provide the *general* solution of the equations of motion (1.3), since they feature the maximal number, *six*, of arbitrary constants, so that this solution can fit *arbitrary* initial data (the three initial positions, and the three initial velocities, of the three particles).

There are some solutions, featuring fewer arbitrary constants, which may be obtained as special, or limiting, cases of this solution (1.4)–(1.8). For instance, for $\beta = \alpha$ (1.7) is clearly replaced by

$$\zeta_{\pm}(\tau) = a\cos(2\lambda\tau + 2\alpha \pm \pi/3) + b\cos(\lambda\tau - \alpha \mp \pi/3)\exp(-\sqrt{3}\lambda\tau). \quad (1.9)$$

As can be readily verified, another special solution, which also features (only) five arbitrary constants, $z_1(0), z_2(0), z_3(0), V, a$, reads

$$z_1(t) = z_1(0) + \tfrac{3}{2}(V + w)\tau + \tfrac{1}{2}a\tau^2$$
$$z_2(t) = z_2(0) + \tfrac{3}{2}(V - w)\tau - \tfrac{1}{2}a\tau^2 \qquad z_3(t) = z_3(0) \quad (1.10a)$$

with

$$w^2 = 2V^2 + \tfrac{4}{9}a[z_1(0) - z_2(0)]. \quad (1.10b)$$

Analogous solutions may of course be obtained by permutation of the particles.

And, as can be readily verified, another solution is given by (1.4) with (1.5), but with (1.6) and (1.7) replaced by

$$Z(\tau) = Z(0) \quad (1.11)$$

$$\zeta_{\pm}(\tau) = \pm a\exp(\alpha\tau \pm \mathrm{i}\pi/6) \quad (1.12)$$

it features only three arbitrary constants: $Z(0), a, \alpha$.

In section 2 we discuss the motion of the three particles in the plane, as described by these solutions. In section 3 we explain how the general solution (1.4)–(1.8) has been arrived at.

2. Discussion

Firstly, let us report [1–3] the Hamiltonian that yields the equations of motion (1.3):

$$H(\underline{p}, \underline{q}) = \sum_{n=1}^{3}\left\{\mathrm{i}(\omega/k)q_n + \exp(kp_n)\left[\prod_{m=1, m\neq n}^{3}(q_n - q_m)\right]^{1/2}\right\}. \quad (2.1)$$

Here k is an arbitrary (nonvanishing, possibly complex) constant, and, as can be readily verified, the (complex version (1.3) of the) equations of motion of our model are obtained in the standard manner from this Hamiltonian, via the identification $q_n = z_n$. A real Hamiltonian written in terms of the real 2-vectors $\vec{r}_n$ and yielding the real equations of motion in the plane, see (1.1), can of course be obtained from (2.1) in the standard manner [2].

Secondly, let us point out that, as immediately implied by (1.3), the centre-of-mass

$$\bar{z}(t) = [z_1(t) + z_2(t) + z_3(t)]/3 \quad (2.2a)$$

satisfies the linear equation of motion

$$\ddot{\bar{z}}(t) - \mathrm{i}\omega\dot{\bar{z}}(t) = 0 \quad (2.2b)$$

hence its time-evolution is given by the formula

$$\bar{z}(t) = Z(\tau) \quad (2.2c)$$

with (1.5) and (1.6).

Thirdly, let us point out that relation (1.5) among the real 'physical' time t and the complex time-like variable $\tau \equiv \tau(t)$ clearly entails $\tau(0) = 0$, $\dot{\tau}(0) = 1$. Hence the solution (1.4)–(1.8) is characterized by the following relations among the six constants $Z(0), V, a, b, \alpha, \beta$ and the initial data:

$$z_1(0) = Z(0) + \zeta_+(0) \qquad z_2(0) = Z(0) + \zeta_-(0) \qquad z_3(0) = Z(0) - \zeta_+(0) - \zeta_-(0) \tag{2.3a}$$

$$\dot{z}_1(0) = V + \zeta'_+(0) \qquad \dot{z}_2(0) = V + \zeta'_-(0) \qquad \dot{z}_3(0) = V - \zeta'_+(0) - \zeta'_-(0) \tag{2.3b}$$

$$\bar{z}(0) = Z(0) \tag{2.3c}$$

$$\zeta_\pm(0) = a\cos(2\alpha \pm \pi/3) + b\cos(-\beta \mp \pi/3) \pm 2(a^2/b)\sin[2(\alpha-\beta)]\cos(2\alpha+\beta\pm\pi/6) \tag{2.3d}$$

$$\zeta'_\pm(0) = -\lambda\{2a\sin(2\alpha \pm \pi/3) + b\sin(-\beta \mp \pi/3) + \sqrt{3}\cos(-\beta \mp \pi/3) \pm 2(a^2/b)\sin[2(\alpha-\beta)][\sin(2\alpha+\beta\pm\pi/6) - \sqrt{3}\cos(2\alpha+\beta\pm\pi/6)]\} \tag{2.3e}$$

with the constant λ always defined by (1.8). Analogous formulae for the special solutions reported above, see (1.9)–(1.12), are easily obtained.

Fourthly, let us note that the general solution (1.4)–(1.8) can be generally represented as the *linear* superposition of six exponentials in the complex variable τ. The explanation of this remarkable fact is given in the following section.

Let us now provide a terse description of the behaviour of this three-body problem in the plane.

If the parameter ω is *positive*, $\omega > 0$, the behaviour is exceedingly simple: *all* solutions are *completely periodic* with period $T = 2\pi/\omega$. This is clear from the fact that the general solution (1.4)–(1.8) is an *entire* function of (only) the (complex, time-like) variable τ, which is itself a periodic function of the real 'time' variable t, see (1.5). This model provides therefore, in this case, another example of 'nonlinear harmonic oscillators' [6].

If instead the parameter ω vanishes, $\omega = 0$, the behaviour is richer: the rest of this section is devoted to this case. Note that now the variable τ coincides with the time t, $\tau = t$, see (1.5). In this case the centre-of-mass moves as a free particle (linearly, with speed V, see (1.6)). It is moreover clear that the generic solution, see ((1.4), (1.6)–(1.8)), is unbounded: generally, as $t \to \infty$, the three particles spiral outwardly to infinity with speeds that increase exponentially with time at a rate characterized by the constant

$$\rho = \text{Max}\{\sqrt{3}|\text{Re}[\lambda]| + |\text{Im}[\lambda]|, 2|\text{Im}[\lambda]|\} \tag{2.4}$$

see (1.7) and (1.8). There are, however, special solutions which remain confined for all (future) times.

One such case is given by the special solution (1.4), (1.6), (1.8) and (1.9), provided the constant λ is (*real* and) *positive*, $\lambda > 0$. Then clearly this solution features a *limit cycle*, which is approached exponentially in time (with rate $\sqrt{3}\lambda$) and is *completely periodic*, with period π/λ.

Another such case is given by the special solution (1.4), (1.11) and (1.12), provided the constant α is purely imaginary, say $\alpha = \mathrm{i}\Omega$ with Ω real. Then clearly this solution, the centre-of-mass of which does not move, see (2.2*c*) and (1.11), is *completely periodic* with period $2\pi/\Omega$.

Finally, a solution worth noting is (1.10), in which one particle does not move, while the other two clearly move as if they were in the presence of a constant force, acting on them with opposite signs. But because the standing particle does not influence the motion of the other two particles (see (1.3)), this solution corresponds in fact rather to a two-body, than a three-body, problem.

3. Proof and history

Since the explicit formulae (1.4)–(1.8) provide the general solution of the equations of motion (1.3), one could dispense with any additional discussion and simply require the diligent reader to verify that (1.4)–(1.8) indeed satisfy (1.3). This would be inappropriate on two counts: it would not explain how this solution has been discovered, and it would omit to allocate the due credit for a remark that was essential to obtain this result. So let us provide a terse history of the developments that led to the solution (1.4)–(1.8).

The first step is to recall [1–3] that, via the change of independent variable

$$z_n(t) = \zeta_n(\tau) \tag{3.1}$$

with the new independent variable τ related to the time t by (1.5), the equations of motion (1.2*a*) take the form

$$\zeta_n'' = 2 \sum_{m=1, m\neq n}^{N} a_{nm} \zeta_n' \zeta_m' / (\zeta_n - \zeta_m). \tag{3.2}$$

Here, and throughout this paper, appended primes denote differentiations with respect to the independent variable τ.

The analytic structure of the solutions $\zeta_n = \zeta_n(\tau)$ of this system of coupled ODEs, (3.2), is of much interest, both as a mathematical problem in its own right and because it clearly determines, via (1.5), the behaviour of the many-body problem in the plane (1.1) [1–3]. This structure depends on the number N of dependent variables ζ_n and on the values of the coupling constants a_{nm} [3]. In particular, in the 'three-body' case ($N = 3$), there are three nontrivial assignments of the coupling constants a_{nm} for which *all* solutions $\zeta_n = \zeta_n(\tau)$ of (3.2) were expected to be *entire* functions of the independent variable τ [7], a property that we called [7] 'super-Painlevé'—in analogy to the standard terminology that attributes the 'Painlevé property' to any (autonomous) nonlinear ODE *all* solutions of which are *meromorphic* functions of the independent variable.

The first of these three cases corresponds to the assignment of coupling constants $a_{12} = a_{21} = 0$, $a_{23} = a_{32} = a_{31} = a_{13} = -1/2$ (of course, up to permutations of the indices). Then the equations of motion (3.2) can be solved explicitly and the general solution turns out to be extremely simple, indeed the functions $\zeta_n(\tau)$ are just third-degree polynomials in the independent variable τ [7].

The second of these three cases corresponds to the assignment of coupling constants $a_{12} = a_{21} = 0, a_{23} = a_{32} = -1/2, a_{31} = a_{13} = -1$ (again, up to permutations). Then from the system of three coupled ODEs (3.2), the following two versions (related to each other by nontrivial differential substitutions) of a single third-order nonlinear ODE were obtained [7]:

$$(y''')^2(y' + V) - y'''(y'')^2 - ky''[3y''y - 2(y' + V)(y' - 2V)] = 0 \qquad y \equiv y(\tau) \tag{3.3}$$

$$\begin{aligned} 8(x''')^2(x' + V)^2(x' - 2V) + 2x'''[-(x'')^2(7x' - 8V) + 9kx'(x' + V)x](x' + V) \\ + 3(x'')^4(2x' - V) - 3k(x'')^2(x' + V)(5x' + 2V)x \\ - 2kx''(x' + V)^2(x' + 4V)^2 + 9k^2(x' + V)^3x^2 = 0 \qquad x \equiv x(\tau). \end{aligned} \tag{3.4}$$

Each of these two third-order autonomous ODEs features the two *arbitrary* constants V, k, which—whenever they do not vanish—could clearly be replaced by unity via a 'cosmetic' rescaling of (dependent and independent) variables. For the reasons stated above, it is expected that each of these two third-order nonlinear ODEs is endowed with the super-Painlevé property [7].

In a previous (version of this) paper [8] we considered the third case identified in [7], namely just the special case characterized by the parameters $N = 3$, $a_{nm} = -1/2$ considered in this paper, see (1.3), to which, via (3.1) with (1.5), there correspond the following equations of type (3.2):

$$\zeta_n'' = -\sum_{m=1,m\neq n}^{3} \zeta_n'\zeta_m'/(\zeta_n - \zeta_m). \tag{3.5}$$

Then, by proceeding in close analogy to [7], we obtained [8] the following single third-order nonlinear autonomous ODE, which we therefore claimed [8] should possess the super-Painlevé property, namely only possess solutions that are *entire* functions of the independent variable τ:

$$4(z''')^3(z' + V) - 3(z'''z'')^2 + k[3z''z - 2(z' + V)(z' - 2V)]^2 = 0 \qquad z \equiv z(\tau). \tag{3.6}$$

Also this ODE features the two *arbitrary* constants V, k, which—whenever they do not vanish—could clearly be replaced by unity via a 'cosmetic' rescaling of (dependent and independent) variables. We also noted [8] that, if $V = 0$, by setting

$$z(\tau) = \exp\left[\int^{\tau} d\tau' u(\tau')\right] \tag{3.7a}$$

this *third-order* autonomous ODE, (3.6), gets reduced to the following *second-order* autonomous ODE:

$$(u'' + 3u'u + u^3)^2(4u''u + 6u'u^2 - 3u'^2 + u^4) + k(3u' + u^2)^2 = 0 \qquad u \equiv u(\tau). \tag{3.7b}$$

And we also pointed out [8] that another avatar of the ODE (3.6) is obtained by differentiating it with respect to the independent variable τ and by then using (3.6) to eliminate the constant k. The advantage of the resulting *fourth-order* autonomous ODE is that it reads

$$z'''' = R(z''', z'', z', z; V) \qquad z \equiv z(\tau) \tag{3.8a}$$

with $R(z''', z'', z', z; V)$ a *rational* function of all its five arguments:

$$R(z''', z'', z', z; V) = z'''[2z'''z - z''(z' - 2V)][3z''z - 2(z' + V)(z' - 2V)]^{-1}. \tag{3.8b}$$

The paper [8] reporting these findings and their proof was not published and is now superseded by the present one, which provides the *general* solution of the equations of motion (3.5), confirming indeed that the solutions of the nonlinear ODEs (3.6) and (3.8) are *entire* (albeit relatively trivially so, see below).

The starting point of our analysis is (3.5), which can be rewritten as follows [7]:

$$\zeta_\pm'' = -(\zeta_+' + V)(\zeta_-' + V)/(\zeta_\pm - \zeta_\mp) + (\zeta_\pm' + V)(\zeta_+' + \zeta_-' - V)/(2\zeta_\pm + \zeta_\mp) \tag{3.9}$$

by setting, consistently with the notation of section 1,

$$\zeta_+(\tau) = \zeta_1(\tau) - Z(\tau) \qquad \zeta_-(\tau) = \zeta_2(\tau) - Z(\tau) \tag{3.10}$$

where $Z(\tau)$ is the centre-of-mass coordinate,

$$Z(\tau) = [\zeta_1(\tau) + \zeta_2(\tau) + \zeta_3(\tau)]/3 \tag{3.11}$$

hence it evolves linearly, see (1.6).

Note that (3.9) represents two (coupled) ODEs, which are obtained by taking systematically the upper, respectively the lower, sign whenever a double sign appears, and which will be hereafter referred to as (upp3.9), respectively (low3.9). This compact way of writing the two equations (3.9) was not used in [8], where a different notation was (unfortunately) used; had we used this more symmetrical notation we might not have missed the crucial observation, see below, which emerges as a consequence of the symmetry among the ODEs satisfied by ζ_+ and ζ_-.

Associated with the system of two ODEs (3.9) is the constant of motion [7]

$$K = (\zeta_+' + V)(\zeta_-' + V)(\zeta_+' + \zeta_-' - V)(\zeta_+ - \zeta_-)^{-1}(\zeta_+ + 2\zeta_-)^{-1}(\zeta_- + 2\zeta_+)^{-1}. \tag{3.12}$$

We now solve (upp3.9) for ζ_-', insert the resulting expression in (3.12) and thereby get, after some algebra, the following expression for K:

$$K = -[4(\zeta_+ + 2\zeta_-)^3(\zeta_+' + V)]^{-1}\{[\zeta_+(\zeta_+ + 2\zeta_-)]^2 - [3\zeta_+''\zeta_+ - 2(\zeta_+' + V)(\zeta_+' - 2V)]^2\}. \tag{3.13}$$

Next, we differentiate (upp3.9) with respect to the independent variable τ, then use (low3.9) to eliminate ζ_-'' (on the right-hand side), then, as above, use (upp3.9) to eliminate ζ_-' (on the right-hand side), and we thereby obtain (after some cumbersome but straightforward computations) the following expression for ζ_+''':

$$\zeta_+''' = 3[4(\zeta_+ + 2\zeta_-)^2(\zeta_+' + V)]^{-1}\{[\zeta_+''(\zeta_+ + 2\zeta_-)]^2 - [3\zeta_+''\zeta_+ - 2(\zeta_+' + V)(\zeta_+' - 2V)]^2\}. \tag{3.14}$$

Now a comparison of (the right-hand sides of) (3.13) with (3.14) yields the simple (linear!) relation

$$\zeta_+''' = -3K(\zeta_+ + 2\zeta_-). \tag{3.15a}$$

The insertion of the expression of the combination $\zeta_+ + 2\zeta_-$ entailed by this formula, (3.15*a*), on the right-hand side of (3.14) yields the third-order ODE (3.6) with

$$z(\tau) \equiv \zeta_+(\tau) \tag{3.16a}$$

$$k = 27K^2. \tag{3.16b}$$

This ends the derivation of (3.6), as reproduced from [8].

But at this point a crucial—and *a posteriori* obvious—remark becomes relevant (the credit for which is given in the acknowledgments section): given the evident symmetry of the system of two coupled ODEs (3.9) under the exchange of ζ_+ with ζ_-, and the no less evident antisymmetry under such an exchange of definition (3.12) of K, it is clear that the following equation analogous to (3.15*a*) also holds:

$$\zeta_-''' = 3K(\zeta_- + 2\zeta_+). \tag{3.15b}$$

And it is then clear (see below) that the two ODEs (3.15) entail that each of the two unknown functions $\zeta_+(\tau)$ and $\zeta_-(\tau)$ satisfies a very simple, decoupled, sixth-order *linear* ODE, the general solution of which is a linear combination of six exponentials (featuring six arbitrary constant coefficients). This is, of course, sufficient to conclude, see (3.16*a*), that the solution of the nonlinear third-order ODE (3.6) must have the same form; hence this nonlinear ODE does indeed possess the super-Painlevé property, but rather trivially so (in particular, it does not define a new transcendental function). However, the solution of this third-order nonlinear ODE, (3.6), can only contain three arbitrary constants; hence, the general solution of the sixth-order linear equation satisfied by $z(t)$ shall also satisfy the third-order nonlinear ODE (3.6) only if its six coefficients satisfy three relations, so that only three of them can take arbitrary values. The task of obtaining these relations is in principle trivial, but in practice quite cumbersome.

We are actually interested in an analogous task, since our main purpose now is to solve the system of two coupled nonlinear ODEs (3.9) for the two unknown functions $\zeta_+(\tau)$ and $\zeta_-(\tau)$, the general solution of which must contain four arbitrary constants. This task is also trivial in principle, but in practice it is sufficiently hard to justify providing here a brief record of how it was achieved.

The two linear ODEs (3.15) were summed and subtracted to yield

$$\sigma''' = 3K\delta \qquad \delta''' = -9K\sigma \tag{3.17}$$

where we introduce conveniently the sum and difference of the two unknowns ζ_+ and ζ_-,

$$\sigma(\tau) = \zeta_+(\tau) + \zeta_-(\tau) \qquad \delta(\tau) = \zeta_+(\tau) - \zeta_-(\tau). \tag{3.18}$$

The two ODEs (3.17) of course entail

$$\sigma'''''' = -k\sigma \qquad \delta'''''' = -k\delta \tag{3.19}$$

with the constant k defined by (3.16b). The general solution of these two decoupled sixth-order linear ODEs reads

$$\sigma(\tau) = \sum_{j=1}^{6} s_j \exp\{2\lambda \exp[\mathrm{i}(2j+1)/6]\tau\} \tag{3.20a}$$

$$\delta(\tau) = \sum_{j=1}^{6} d_j \exp\{2\lambda \exp[\mathrm{i}(2j+1)/6]\tau\} \tag{3.20b}$$

with

$$(2\lambda)^6 = k \tag{3.21}$$

(this notation is chosen for consistency with that used in section 1, see (1.4)), and it clearly also entails, via (3.18),

$$\zeta_\pm(\tau) = \frac{1}{2}\sum_{j=1}^{6} (s_j \pm d_j) \exp\{2\lambda \exp[\mathrm{i}(2j+1)/6]\tau\}. \tag{3.22}$$

Here the 12 constants s_j, d_j are *a priori* arbitrary, but simple relations among them are obviously entailed by the ODEs (3.17). Using these relations it is easy to arrive at the following expressions for the two unknown functions $\zeta_+(\tau)$ and $\zeta_-(\tau)$:

$$\begin{aligned}\zeta_\pm(\tau) = {}& a\cos(2\lambda\tau + 2\alpha \pm \pi/3) + b\cos(\lambda\tau - \beta \mp \pi/3)\exp(-\sqrt{3}\lambda\tau) \\ & + c\cos(\lambda\tau + \gamma \mp \pi/6)\exp(\sqrt{3}\lambda\tau)\end{aligned} \tag{3.23}$$

where the notation has again been adjusted to fit eventually with the solution (1.7) and (1.8).

These expressions, (3.23), now contain the seven, *a priori* arbitrary, constants $a, b, c, \alpha, \beta, \gamma, \lambda$. The final step is to impose that these expressions, (3.23), satisfy the ODEs (3.9), and to thereby determine, see (1.7) and (1.8), the values of the constants c, γ, λ in terms of the others (and of the constant V that appears in (3.9)). This has been achieved by eliminating firstly the denominators in (3.9), then inserting expressions (3.23) in the ODEs so obtained and finally setting $\exp(\mathrm{i}\lambda\tau) = u$, $\exp(\sqrt{3}\lambda\tau) = v$. The resulting expressions are then (after elimination of the denominators) just polynomial in the two quantities u and v, which can then be treated as independent variables. Hence each coefficient of these polynomials must vanish. One can then firstly set to zero the coefficients that appear simpler, obtain thereby explicit expressions for c, γ, λ and then verify that, as it were 'miraculously,' all the other coefficients also vanish. The computations we just described are straightforward but quite cumbersome, and we were only able to perform them with the help of MAPLE. In this manner the solution (1.4)–(1.8) was arrived at.

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Note added in proof. After this paper was completed and submitted for publication it was discovered that the problem characterized by the equations of motion (1.1) and (1.2) can also be solved in the more general N-body 'nearest-neighbour' case when the coupling constants a_{nm} vanish unless $|n - m| = 1$ and equal minus one half otherwise, which reduces for $N = 3$ to the case treated above. For more information on the history of this more general problem and on its solution, the interested reader is referred to a forthcoming publication [9].

References

[1] Calogero F and Françoise J-P 2001 Periodic solutions of a many-rotator problem in the plane *Inverse Probl.* **17** 1–8
[2] Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments (Lecture Notes in Physics Monograph* vol m 66*)* (Berlin: Springer)
[3] Calogero F, Françoise J-P and Sommacal M 2003 Periodic solutions of a many-rotator problem in the plane. II. Analysis of various motions *J. Nonlinear Math. Phys.* **10** 157–214
[4] Calogero F 1978 Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations, and related 'solvable' many-body problems *Nuovo Cimento* B **43** 177–241
[5] Calogero F 2001 The 'neatest' many-body problem amenable to exact treatments (a 'goldfish'?) *Physica* D **152–153** 78–84
[6] Calogero F and Inozemtsev V I 2002 Nonlinear harmonic oscillators *J. Phys. A: Math. Gen.* **35** 10365–75
[7] Calogero F 2002 Solvable three-body problems and Painlevé conjectures *Theor. Math. Phys.* **133** 1443–52
Calogero F 2003 *Theor. Math. Phys.* **134** 139 (erratum)
[8] Calogero F A super-Painlevé third-order ODE unpublished
[9] Calogero F Solution of the goldfish N-body problem in the plane with (only) nearest-neighbor coupling constants all equal to minus one half *J. Nonlinear Math. Phys.* at press